Analysis of stochastic partial differential equation solutions for disease modeling

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ABSTRACT

The main objective of this paper is to study the viscosity technical to solve stochastic partial differential equations(SPDEs) and to discuss the uniqueness of the stochastic viscosity solution and also to study a comparison theorem between a stochastic viscosity solution and an ω-wise stochastic viscosity solution.

Keywords: Stochastic PDEs; Stochastic viscosity solution; Uniqueness; ω-wise stochastic viscosity solution.

I. INTRODUCTION

A stochastic differential equation (SDE for briefly) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. The earliest work on SDEs was done to describe Brownian motion in Einstein's famous paper, and by the physicist Marian Smoluchowski The earlier works done on SDEs related to Brownian motion is credited to Bachelier (1900) in his thesis 'Theory of Speculation. The stochastic differential equations with it is both ordinary and partial types are used to model diverse phenomena. The existence and uniqueness of solution for SDEs found in Øksendal see [1].

Stochastic Partial Differential Equation (SPDEs) are essentially partial differential equation that have random forcing term and coefficients. The most classical SPDEs is given by stochastic heat equation can formally be written as $\partial_t u = \Delta u + \mathcal{F}$ where $\mathcal{F}(t, x)$ is random noise and Δ is the laplacian an $u(t, x), t \geq 0, \ \ 0 \leq x \leq L$.

II. VISCOSITY SOLUTION

The theory of viscosity can be applied to study linear and nonlinear Partial differential equations of any order. We can now introduce the definition of viscosity solutions for PDEs:

$$F(x, u, Du, D^2u) = 0, \ \ x \in \Omega \qquad \dots\dots\dots\dots\dots\dots\dots (1)$$

Definition 2.1: Let $\Omega \subset \mathbb{R}^n$ be an open set and u continuous in Ω.

(i) We say that u is a viscosity of (1) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, then

$$F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \leq 0; \qquad \dots\dots\dots\dots\dots\dots\dots (2)$$

(ii) We say that u is a viscosity super solution of (6) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, then

$$F\big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\big) \geq 0; \qquad \dots\dots\dots\dots\dots\dots\dots (3)$$

(iii) We say that u is a viscosity solution in the open set Ω if u is a viscosity subsolution and a viscosity supersolution, at any point $x_0 \in \Omega$.

III. PROBLEM FORMULATION

We continue to study the following nonlinear stochastic PDE (SPDE):

$$du(t, x) = \big\{\mathcal{A}u(t, x) + f\big(t, x, u(t, x), \sigma^*(x)Du(t, x)\big)\big\}dt + \sum_{i=1}^{d} g_i\big(t, x, u(t, x)\big)dB_t^i\,(t, x) \in (0, t) \times R^n,$$

$$u(0, x) = u_0(x), \qquad x \in R^n, \qquad \dots\dots\dots\dots\dots\dots (4)$$

Where $B = (B^1, \dots, B^d)$ is a standard d-dimensional Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, p; F)$, with $F = \{\mathcal{F}_t\}_{t\geq 0}$ being a filtration satisfying the usual hypothesis and the stochastic integral in the second-order differential operator $\mathcal{A}$ in (4) is defined by

$$\mathcal{A} = \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)^* D^2) + \langle \beta(x), D \rangle, \qquad \dots\dots\dots\dots(5)$$

Where $D = (\partial_{x_1}, \partial_{x_2}, \dots \partial_{x_n})^T, D^2 = (\partial^2_{x_i x_j})^n_{i,j=1}$; the functions σ, β are assumed to be measurable; and $\sigma^*(.)$ denotes the transpose of $\sigma(.)$. To simplify notations we also denote,

$$F(\omega, t, x, y, p, A) \triangleq \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)^* A) + \langle \beta(x), p \rangle + f(\omega, t, x, y\sigma^*(x)p)$$
$$(\omega, t, x, y, p, A) \in \Omega \times [0,T] \times R^n \times R \times R^n \times \varphi^n, \qquad \dots\dots\dots(6)$$

Where φ^n is the space of all symmetric n ×n-matrices. In the sequel we will refer to (4) as SPDE (f, g).
We shall make use of the following standing assumptions:
(i) The functions $\sigma: R^n \to R^{n\times k}$ and $\beta: R^n \to R$ are uniformly Lipschitz continuous, with a common Lipschitz constant $K > 0$.
(ii) The function $f: \Omega \times [0,T] \times R^n \times R \times R^k \mapsto R$ is a continuous random field such that for fixed $(x, y, p), f(.,.,x, y, \sigma^*(x)p)$ is F^B-progressively measurable; and there exists some constant$K > 0$ such that for P-a.e. $\omega \in \Omega$,

$$|f(\omega, t, x, 0,0)| \leq K \ \ \forall (t, x) \in [0,T] \times R,$$
$$|f(\omega, t, x, y, z) - f(\omega, t', x', y', z')| \leq$$
$$K(|t - t'| + |x - x'| + |y - y'| + |z - z'|);$$
$$\forall (t, x, y, z), (t', x', y', z') \in [0,T] \times R^n \times R \times R^k \ \ \dots\dots(7)$$

(iii) The function $u_o: R^n \mapsto R$ is continuous and, such that for some constants $K, p > 0$,

$$|u_0(x)| \leq K(1 + |x|^p), \quad x \in R^n \qquad \dots\dots\dots\dots\dots\dots\dots(8)$$

(iv) The function $g \in C_b^{0,2,3}([0,T] \times R^n \times R; R^d)$.
Recall that in order to obtain the so-called uniform stochastic boundedness of the stochastic viscosity solution, we need to strengthen Assumption (iv) to the following:
(v) The function g satisfies (iv); and for $\varepsilon > 0$, there exists a function $G^\varepsilon \in C^{1,2,2,2}([0,T] \times R^d \times R^n \times R)$

$$\frac{\partial G^\varepsilon}{\partial t}(t, w, x, y) = \varepsilon; \ \frac{\partial G^\varepsilon}{\partial w^i} = g_i(t, x, G^\varepsilon(t, w, x, y)), \ \ i = 1, \dots, d;$$
$$G^\varepsilon(0,0, x, y) = y \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots(9)$$

We remark that when g is independent of $t$and $d = 1$, then (v) is trivially satisfied, since one can always first solve the ODE (with parameter x):

$$\frac{dG}{dw} = g(x, G), \qquad G(0) = y,$$

And then set $G^\varepsilon(t, w, x, y) = G(w, x, y) + \varepsilon t$.

To define a stochastic viscosity solution, we first consider the following SDE in the sense: for each $(x, y) \in R^n \times R$,

$$\eta(t, x, y) = y + \sum_{i=1}^{d} \int_0^t g_i(s, x, \eta(s, x, y))\, dB_s^i$$
$$\triangleq y + \int_0^t \langle g(s, x, \eta(s, x, y)) \rangle, \ \ t \geq 0, \qquad \dots\dots\dots\dots(10)$$

or equivalently, an $It\hat{o}$ SDE (with parameter)

$$\eta(t, x, y) = y + \frac{1}{2}\int_0^t \langle g, D_y g \rangle (s, x, \eta(s, x, y)) ds$$

$$+\int_0^t \langle g(s,x,\eta(s,x,y))dB_s\rangle \qquad \dots\dots\dots\dots\dots (11)$$

Denote the (unique) solution of (10) or (11) by $\eta(t,x,y),(t,x,y) \in [o,T] \times R^n \times R$. From the theory of SDE_s we know that, as a stochastic flow, $\eta \in C(F^B,[o,T] \times R^n \times R)$. Since under (iv) the mapping $y \mapsto \eta(t,x,y,\omega)$ defines a diffeomorphism for all (t,x), a.s., we can denote the y-inverse of $\eta(t,x,y)$ by $\boldsymbol{\varepsilon}(t,x,y)$, and show that $\boldsymbol{\varepsilon}(t,x,y)$ is the solution to the following SPDE:

$$\boldsymbol{\varepsilon}(t,x,y) = y - \int_0^t \langle D_y \boldsymbol{\varepsilon}(t,x,y) g(s,x,y), dB_s\rangle$$

$$\forall (t,x,y), a.s. \qquad \dots\dots\dots\dots\dots\dots (12)$$

Definition 3.1 A random field $u \in C(F^B,[0,T] \times R^n)$ is called a stochastic viscosity subsolution (resp. supersolution) of SPDE (f,g), if $u(0,t) \le (resp.) u_0(x) \forall x \in R^n$; and if for any $\tau \in \mathcal{M}_0^B, \xi \in L^0(\mathcal{F}_\tau^B; R^n)$, and any random field $\varphi \in C^{1,2}(\mathcal{F}_\tau^B,[o,T] \times R^n)$ satisfying.

$u(t,x) - \eta(t,x,\varphi(t,x)) \le (res. \ge) 0 = u(\tau,\xi) - \eta(\tau,\xi,\varphi(\tau,\xi))$,
Foe all (t,x) in a neighborhood of (τ,ξ), a.e. on the set $\{0 < \tau < T\}$, it holds that

$$\mathcal{A}\psi(\tau,\xi) + f(\tau,\xi,\psi(\tau,\xi),\sigma^*(\xi)D\psi(\tau,\xi)) \ge (resp. \le) D_y\eta(\tau,\xi,\varphi(\tau,\xi)D_t\varphi(\tau,\xi)),$$

a.e. on $\{0 < \tau < T\}$, where $\psi(t,x) \triangleq \eta(t,x,\varphi(t,x))$ $\dots\dots\dots\dots (13)$

A random field $u \in C(F^B,[0,T] \times R^n)$ is called a stochastic viscosity solution of SPDE(f,g), if it is both a stochastic viscosity subsolution and a supersolution.

In the special case when $g = 0$, one can view SPDE $(f,0)$ as a PDE with random coefficients. Therefore, for each $\omega \in \Omega$ one can define the viscosity solution to SPDE $(f,0)$ in the deterministic sense. Taking the ω-measurability into account we have the following definition which is important for the study of uniqueness.

Definition 3.2: A random field $u \in C(F^B,[0,T] \times R^n)$ is called an ω-wise viscosity (sub, super) solution if for a.e. $\omega \in \Omega, u(\omega,.,.)$ is a (deterministic) viscosity (sub, super) solution of the SPDE $(f,0)$.

Definition 3.3: A random field $u \in C(F^B,[0,T] \times R^n)$ is called stochastically uniformly bounded if there exists a positive, increasing process $\Theta \in L^0(F^B,[0,T])$, such that P-almost surely, it holds that $|u(t,x)| \le \Theta_t \forall (t,x) \in [0,T] \times R^n$.

Theorem 3.4: Assume (i)-(iv). A random field u is a stochastic viscosity sub (resp. super) solution to SPDE (f,g) if and only if $v(.,.) = \varepsilon(.,.,u(.,.))$ is a stochastic viscosity sub (resp. super) solution to SPDE $(\tilde{f},0)$.

Consequently, u is a stochastic viscosity of SPDE (f,g) if and only if $v(.,.) = \varepsilon(.,.,u(.,.))$ is a stochastic viscosity solution SPDE $(\tilde{f},0)$.
Theorem 3.5: Assume (i)-(iv). Then the SPDE (f,g) admits a stochastic viscosity solution $u \in C(F^B,[0,T] \times R^n)$; and SPDE $(\tilde{f},0)$admits a stochastic viscosity solution $v \in C(F^B,[0,T] \times R^n)$. Furthermore, a pair of these solutions u,v can be related as
$u(t,x) = \eta(t,x,v(t,x))$; $v(t,x) = \varepsilon(t,x,u(t,x))$,

Where η and ε are the solutions to (11) and (12), respectively.

Finally, if in addition (v) holds and u_0 is uniformly bounded, then the random fields u and v are both stochastically uniformly bounded.

The uniqueness results are contained in the following corollary. Recall that in the case of $g = 0$, an ω-wise viscosity is necessary stochastic viscosity solution.

Corollary 3.6 Assume (i) – (v). Then

i) If $v_1 \in C(F^B, [0,T] \times \mathbb{R}^n)$ is a stochastic viscosity solution and $v_2 \in C(F^B, [0,T] \times \mathbb{R}^n)$ is an ω-wise viscosity solution of (3.1), and both are uniformly stochastically bounded, then $v_1(t,x) \equiv v_2(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$, P-a.s.

ii) The uniformly stochastically boundedω-wise viscosity solution to (3.1) is unique. In particular, if $\tilde{f}$ is deterministic, then the uniformly bounded; deterministic viscosity solution of(3.1) is unique.

iii) If in addition $(A4')$ also holds, then the stochastic viscosity solution to SPDE (f,g) is unique among uniformly stochastically bounded random fields in$C(F^B, [0,T] \times \mathbb{R}^n)$.

Proof: see in(2)

IV. CONCLUSION AND RESULT

In this paper we introduce a de7nition of stochastic viscosity solution and we show that a stochastic PDE can be converted to a PDE with random coefficients and the uniqueness of the stochastic viscosity solution, where the relation between the stochastic viscosity solution and the ω-wise, "deterministic" viscosity solution to the PDE with random coefficients.

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